THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2023 Enrichment Programme for Young Mathematics Talents
SAYT1134 Towards Differential Geometry
Solution to Test 1 on 16 Aug 2023

1. Let $\mathbf{u} = (1, 3, 2)$, $\mathbf{v} = (3, 1, 1)$, $\mathbf{z} = (1, 1, 0)$.
   (a) (*4 points*) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{z}$ are linearly independent.
   (b) (*4 points*) Find the area of a parallelogram $OABC$ in $\mathbb{R}^3$ with $\overrightarrow{OA} = \mathbf{u}$ and $\overrightarrow{OC} = \mathbf{v}$.
   (c) (*4 points*) Find the distance between the point $\mathbf{z}$ and the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

Solution.

(a) Suppose there exist scalars $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1\mathbf{u} + a_2\mathbf{v} + a_3\mathbf{z} = \mathbf{0} \ .$$

By comparing coordinate entries, it follows that

$$\begin{cases} a_1 + 3a_2 + a_3 = 0 \\ 3a_1 + a_2 + a_3 = 0 \\ 2a_1 + a_2 \qquad = 0 \end{cases} \tag{1}$$

From the last equation, we have $a_2 = -2a_1$.
Putting into the first and the second equations, we have

$$\begin{cases} -5a_1 + a_3 = 0 \\ a_1 + a_3 = 0 \end{cases}$$

From the difference of these two equations, we have $a_1 = 0$.
It follows that $a_3 = -a_1 = 0$ and $a_2 = -2a_1 = 0$.
Thus, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{z}$ are linearly independent . □

Remark. We can write the system (1) in a matrix representation as:

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

Then, the vectors $\mathbf{u}, \mathbf{v}, \mathbf{z}$ are linearly independent if and only if the homogeneous system (2) has only the trivial solution $a_1 = a_2 = a_3 = 0$, if and only if the coefficient matrix is non-singular if and only if $\det(\mathbf{u}, \mathbf{v}, \mathbf{z}) \neq 0$.

(b) Note that

$$\overrightarrow{OA} \times \overrightarrow{OC} = \mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} \right) = (1, 5, -8)$$

Thus, the area of the parallelogram $OABC$ required is

$$\|\overrightarrow{OA} \times \overrightarrow{OC}\| = \sqrt{1^2 + 5^2 + 8^2} = 3\sqrt{10} \text{ square units.}$$

(c) From (b), note that

$$\langle \mathbf{z}, \mathbf{u} \times \mathbf{v} \rangle = 1(1) + 1(5) + 0(-8) = 6 \quad > 0$$

Thus, the distance between the point $\mathbf{z}$ and the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathrm{proj}_{\mathbf{u}\times\mathbf{v}}(\mathbf{z})\| = \left\| \frac{\langle \mathbf{z}, \mathbf{u} \times \mathbf{v} \rangle}{\|\mathbf{u} \times \mathbf{v}\|^2} \mathbf{u} \times \mathbf{v} \right\| = \frac{\left| \langle \mathbf{z}, \mathbf{u} \times \mathbf{v} \rangle \right|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{6}{3\sqrt{10}} = \frac{\sqrt{10}}{5} .$$

2. This question is related to properties of some special 3×3 matrices.

(a) (*6 points*) Evaluate the following determinants:

$$\det(R_z) = \det \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(R_y) = \det \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\det(R_x) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

(b) i. (*3 points*) Describe the geometric meaning of the three matrices R_x, R_y, R_z above.

ii. (*2 points*) Describe the geometric meaning of the product $R_xR_yR_z$ of the three matrices above.

(c) (*2 points*) Given fixed α, β, γ, someone claims that $R_xR_yR_z = R_zR_xR_y$. Do you agree? Explain.

Solution.

(a) For $\det(R_z)$, spanning along the last row (or last column), we have

$$\det(R_z) = (-1)^{3+3} \cdot 1 \cdot \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

For $\det(R_y)$, spanning along the second row (or second column), we have

$$\det(R_y) = (-1)^{2+2} \cdot 1 \cdot \begin{vmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{vmatrix} = \cos^2\beta + \sin^2\beta = 1$$

For $\det(R_x)$, spanning along the first row (or first column), we have

$$\det(R_x) = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{vmatrix} = \cos^2\gamma + \sin^2\gamma = 1$$

(b) i. R_x represents a proper rotation by an angle γ about the x-axis,
R_y represents a proper rotation by an angle β about the y-axis,
R_z represents a proper rotation by an angle α about the z-axis .

ii. The product $R_xR_yR_z$ represents firstly rotated by an angle α along the z-axis, followed by rotated by an angle β about the y-axis and lastly rotated by an angle γ about the x-axis, with orientation preserving.

(c) No. Putting $\beta = 0$ so that $R_y = I$. It suffices to check that $R_xR_z \neq R_zR_x$ in general by considering **one** of those entries is difference.

Note that

$$[R_xR_z]_{13} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

and

$$[R_zR_x]_{13} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\sin\gamma \\ \cos\gamma \end{bmatrix} = \sin\alpha\sin\gamma \neq 0$$

for some α, γ. This shows that

$$R_xR_z \neq R_zR_x$$

in general and thus the equality

$$R_xR_yR_z = R_zR_xR_y$$

does not hold in general, the claim is disagreed.

3. Let $\gamma(t) = (t, t^2)$ be a curve on the xy-plane.

(a) (*3 points*) Find $\gamma'(t)$ and describe its geometric meaning.

(b) (*5 points*) Find the length of arc in $\gamma(t)$ joining the points $(0, 0)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$.

(c) (*5 points*) Let $\alpha(t) = (-t+3, t^2 - 6t + 9)$. Using the result of (b), find the length of arc in $\alpha(t)$ joining the points $(3, 9)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$. You need to explain very clearly how part (b) helps you find the answer.

Solution.

(a) $\gamma'(t) = (1, 2t)$, and it represents the direction of the tangent vector to the curve $\gamma(t)$.

(b) From (a), note that

$$\|\gamma'(t)\| = \sqrt{1+4t^2}$$

$$\gamma(t) = (0,0) \iff t = 0, \text{ and } \gamma(t) = (\tfrac{3}{2}, \tfrac{9}{4}) \iff t = \tfrac{3}{2}$$

Hence, we have

$$\int_0^{\frac{3}{2}} \|\gamma'(t)\| dt = \int_0^{\frac{3}{2}} \sqrt{1+4t^2}\, dt$$

Putting $2t = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, it follows that

$$t = 0 \iff \theta = 0$$

and

$$t = \frac{3}{2} \iff \theta = \tan^{-1} 3$$

and hence

$$\begin{aligned}
\int_0^{\frac{3}{2}} \sqrt{1+4t^2}\, dt &= \frac{1}{2}\int_0^{\tan^{-1}3} \sec^3\theta\, d\theta \\
&= \frac{1}{2}\int_0^{\tan^{-1}3} \sec\theta \cdot d(\tan\theta) \\
&= \frac{1}{2}\left[\sec\theta\tan\theta\right]_0^{\tan^{-1}3} - \frac{1}{2}\int_0^{\tan^{-1}3} \tan\theta\cdot\sec\theta\tan\theta\, d\theta \\
&= \frac{1}{2}(3\sqrt{10}) - \frac{1}{2}\int_0^{\tan^{-1}3} \sec^3\theta\, d\theta + \frac{1}{2}\int_0^{\tan^{-1}3} \sec\theta\, d\theta \\
&= \frac{3}{2}\sqrt{10} - \frac{1}{2}\int_0^{\tan^{-1}3} \sec^3\theta\, d\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|_0^{\tan^{-1}3} \\
&= \frac{3}{2}\sqrt{10} - \frac{1}{2}\int_0^{\tan^{-1}3} \sec^3\theta\, d\theta + \frac{1}{2}\ln(3+\sqrt{10})
\end{aligned}$$

Thus, the arc length required is given by

$$\int_0^{\frac{3}{2}} \sqrt{1+4t^2}\, dt = \frac{1}{2}\left(\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln(3+\sqrt{10})\right) = \frac{1}{4}(3\sqrt{10} + \ln(3+\sqrt{10})).$$

Remark. If you use hyperbolic substitution, the alternative answer is given by

$$\frac{1}{4}(\sinh^{-1}3 + 3\sqrt{10}).$$

(c) Observe that

$$\alpha(t) = \gamma(3-t) \qquad (*)$$

Also, we have

$$\alpha(t) = (3, 9) \iff t = 0$$

and

$$\alpha(t) = (\tfrac{3}{2}, \tfrac{9}{4}) \iff t = \tfrac{3}{2}$$

It follows that

$$\alpha(0) = (3, 9) = \gamma(3)$$

and

$$\alpha(\tfrac{3}{2}) = (\tfrac{3}{2}, \tfrac{9}{4}) = \gamma(\tfrac{3}{2})$$

The length of arc in $\alpha(t)$ joining the points $(3, 9)$ and $(\frac{3}{2}, \frac{9}{4})$ is given by

$$\int_0^{\frac{3}{2}} \|\alpha'(t)\| dt = \int_3^{\frac{3}{2}} \|\gamma'(3-t)\| \cdot (-1) dt$$

Using substitution $u = 3 - t$, then it follows that

$$\int_3^{\frac{3}{2}} \|\gamma'(3-t)\| \cdot (-1) dt = \int_0^{\frac{3}{2}} \|\gamma'(u)\| \cdot (-1)(-du) = \int_0^{\frac{3}{2}} \|\gamma'(u)\| du$$
$$= \frac{1}{4}(3\sqrt{10} + \ln(3 + \sqrt{10}))$$

Thus, the arc-length of $\alpha(t)$ required is

$$\int_0^{\frac{3}{2}} \|\alpha'(t)\| dt = \int_0^{\frac{3}{2}} \|\gamma'(t)\| dt = \frac{1}{4}(3\sqrt{10} + \ln(3 + \sqrt{10}))\ .$$

Remark. It is also acceptable if you write

$$\frac{1}{4}(\sinh^{-1} 3 + 3\sqrt{10})\ .$$

4. Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

(a) (*5 points*) Prove that $\langle \mathbf{u}, \mathbf{v} \rangle = \dfrac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right)$.

(b) (*5 points*) Let A be a 3×3 matrix such that for any $\mathbf{v} \in \mathbb{R}^3$, $\|A\mathbf{v}\| = \|\mathbf{v}\|$. Show that for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

(c) (*5 points*) Write down any 3×3 matrix A except $\pm I$, such that for all $\mathbf{v} \in \mathbb{R}^3$,

$$\|A\mathbf{v}\| = \|\mathbf{v}\|.$$

Solution.

(a) Note that

$$\begin{aligned}\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{u}-\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u}-\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle\end{aligned}$$

Since $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, hence we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

and thus

$$\frac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right) = \frac{1}{2}\left(2\langle \mathbf{u}, \mathbf{v} \rangle\right) = \langle \mathbf{u}, \mathbf{v} \rangle . \qquad \square$$

(b) Observe that

$$\langle A\mathbf{v}, A\mathbf{v} \rangle = (A\mathbf{v})^T(A\mathbf{v}) = \mathbf{v}^T(A^TA\mathbf{v}) = \left\langle \mathbf{v}, A^TA\mathbf{v} \right\rangle = \langle \mathbf{v}, \mathbf{v} \rangle$$

it follows that

$$\mathbf{v} - A^TA\mathbf{v} = \mathbf{0} \implies A^TA\mathbf{v} = \mathbf{v}$$

for all $\mathbf{v} \in \mathbb{R}^3$.

Thus, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we have

$$\langle A\mathbf{u}, A\mathbf{v} \rangle = (A\mathbf{u})^T(A\mathbf{v}) = \mathbf{u}^T(A^TA\mathbf{v}) = \mathbf{u}^T\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle . \qquad \square$$

(c) From part (b), we have

$$A^TA\mathbf{v} = \mathbf{v}$$

for any $\mathbf{v} \in \mathbb{R}^3$. By taking $\mathbf{v} = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_2$ and $\mathbf{v} = \mathbf{e}_3$, it follows that

$$A^TA\left[\mathbf{e}_1 \,|\, \mathbf{e}_2 \,|\, \mathbf{e}_3\right] = \left[A^TA\mathbf{e}_1 \,|\, A^TA\mathbf{e}_2 \,|\, A^TA\mathbf{e}_3\right] = \left[\mathbf{e}_1 \,|\, \mathbf{e}_2 \,|\, \mathbf{e}_3\right] = I$$

Therefore, it shows that $A^TA = I$, i.e. A is an orthogonal matrix.

If $A = [\mathbf{a}_1 \,|\, \mathbf{a}_2 \,|\, \mathbf{a}_3]$, where $\mathbf{a}_i$ is the i-th column vector of the matrix A for $i = 1, 2, 3$.

Then, we have

$$A^TA = I \iff \mathbf{a}_i^T\mathbf{a}_j = \left\langle \mathbf{a}_i, \mathbf{a}_j \right\rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

for $1 \leq i, j \leq 3$. It shows that any 3×3 matrices A with column vectors forming an orthonormal set will satisfy

$$\|A\mathbf{v}\| = \|\mathbf{v}\| .$$

Example: Except for $\pm I$, any 3×3 matrices with column vectors forming an orthonormal set is acceptable.

5. (a) Let $\mathbf{u} : (-\pi, \pi) \to \mathbb{R}^2$ be such that

$$\mathbf{u}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{u}(t)$$

and $\mathbf{u}(0) = \mathbf{e}_1$.

i. (*2 points*) Show that $\langle \mathbf{u}'(t), \mathbf{u}(t) \rangle = 0$ for any $t \in (-\pi, \pi)$.

ii. (*1 point*) By (i), show that $\dfrac{d}{dt}\left(\|\mathbf{u}\|\right) = 0$ for any $t \in (-\pi, \pi)$.

iii. (*1 point*) By (ii), show that $\|\mathbf{u}\| = 1$ for any $t \in (-\pi, \pi)$.

(b) It is given that $\left\langle \mathbf{u}(t), \mathbf{e}_2 \right\rangle = \sin t$ for any $t \in (-\pi, \pi)$.

i. (*2 points*) Find explicit formula for $\mathbf{u}(t)$.

ii. (*1 point*) What shape does $\mathbf{u}(t)$ trace?

iii. (*3 points*) Define

$$\kappa_{\mathbf{u}(t)} = \frac{\det(\mathbf{u}'(t), \mathbf{u}''(t))}{\|\mathbf{u}'(t)\|^3} ,$$

show that $\kappa_{\mathbf{u}(t)} = 1$ for any $t \in (-\pi, \pi)$.

iv. (*2 points*) Let $\mathbf{v}(t) = \alpha\, \mathbf{u}(t)$ where $\alpha > 0$ is a constant. Find $\kappa_{\mathbf{v}}$ in terms of α.

Solution.

(a) i. Letting $\mathbf{u}(t) = (\alpha(t), \beta(t))$, where α, β are twice differentiable functions on $(-\pi, \pi)$. Then

$$\mathbf{u}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} -\beta(t) \\ \alpha(t) \end{pmatrix}$$

Thus, it follows that

$$\langle \mathbf{u}'(t), \mathbf{u}(t) \rangle = (\mathbf{u}'(t))^T \mathbf{u}(t) = -\beta(t)\alpha(t) + \alpha(t)\beta(t) = 0 .$$ □

ii. From (i), we have

$$0 = \langle \mathbf{u}'(t), \mathbf{u}(t) \rangle = \frac{1}{2}\frac{d}{dt} \langle \mathbf{u}(t), \mathbf{u}(t) \rangle = \frac{1}{2}\frac{d}{dt} \|\mathbf{u}(t)\|^2$$

This shows that $\|\mathbf{u}(t)\|^2$ is a constant (independent with variable t), hence $\|\mathbf{u}(t)\|$ is also a constant and thus

$$\frac{d}{dt}\left(\|\mathbf{u}\|\right) = 0$$

for any $t \in (-\pi, \pi)$. □

iii. From (a)(ii), let $\|\mathbf{u}(t)\| = c$ for some positive constants c, for any $t \in (-\pi, \pi)$. Since $\mathbf{u}(0) = \mathbf{e}_1$, hence

$$c = \|\mathbf{u}(0)\| = \|\mathbf{e}_1\| = 1$$

and thus

$$\|\mathbf{u}\| = 1$$

for any $t \in (-\pi, \pi)$. □

(b) i. Note that $\left\{\mathbf{e}_1, \mathbf{e}_2\right\}$ constitutes an orthonormal basis for $\mathbb{R}^2$. Therefore, we have

$$\mathbf{u}(t) = \left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{u}(t), \mathbf{e}_2 \right\rangle \mathbf{e}_2 \qquad (*)$$

Since $\|\mathbf{u}(t)\| = 1$ for any $t \in (-\pi, \pi)$, it follows that

$$\left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle^2 + \left\langle \mathbf{u}(t), \mathbf{e}_2 \right\rangle^2 = 1$$

$$\left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle^2 = 1 - \sin^2 t = \cos^2 t$$

Hence, we have

$$\left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle = \cos t \quad \text{or} \quad \left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle = -\cos t$$

Since $\mathbf{u}(0) = \mathbf{e}_1$, hence

$$\left\langle \mathbf{u}(0), \mathbf{e}_1 \right\rangle = \left\langle \mathbf{e}_1, \mathbf{e}_1 \right\rangle = 1 = \cos(0)$$

Therefore $\left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle = -\cos t$ is rejected and hence we have

$$\left\langle \mathbf{u}(t), \mathbf{e}_1 \right\rangle = \cos t\ .$$

Thus, from (*), we have

$$\mathbf{u}(t) = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 = (\cos t,\ \sin t)\ .$$

ii. From (b)(i), we have

$$\|\mathbf{u}(t) - \mathbf{0}\| = 1$$

for any $t \in (-\pi, \pi)$, it clearly shows that $\mathbf{u}(t)$ is a unit circle centered at origin.

iii. Note that

$$\begin{cases} \mathbf{u}'(t) = (-\sin t, \cos t) \\ \mathbf{u}''(t) = (-\cos t, -\sin t) \end{cases}$$

From the formula provided, we have

$$\begin{aligned}
\kappa_{\mathbf{u}(t)} &= \frac{\det(\mathbf{u}'(t), \mathbf{u}''(t))}{\|\mathbf{u}'(t)\|^3} \\
&= \frac{\begin{vmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{vmatrix}}{\left(\sqrt{(-\sin t)^2 + (\cos t)^2}\right)^3} \\
&= \frac{\sin^2 t + \cos^2 t}{\left((\sin t)^2 + (\cos t)^2\right)^{\frac{3}{2}}} \\
&= 1
\end{aligned}$$

for any $t \in (-\pi, \pi)$.

iv. Note that for $\alpha > 0$, then

$$\begin{cases} \mathbf{v}'(t) = \alpha\,\mathbf{u}'(t) \\ \mathbf{v}''(t) = \alpha\,\mathbf{u}''(t) \end{cases}$$

Hence, it follows that

$$\begin{aligned}
\kappa_{\mathbf{v}(t)} &= \frac{\det(\mathbf{v}'(t), \mathbf{v}''(t))}{\|\mathbf{v}'(t)\|^3} \\
&= \frac{\det(\alpha\,\mathbf{u}'(t), \alpha\,\mathbf{u}''(t))}{\|\alpha\,\mathbf{u}'(t)\|^3} \\
&= \frac{\alpha^2 \det(\mathbf{u}'(t), \mathbf{u}''(t))}{\alpha^3\|\mathbf{u}'(t)\|^3} \\
&= \frac{1}{\alpha} \cdot \kappa_{\mathbf{u}(t)} \\
&= \frac{1}{\alpha}
\end{aligned}$$

for any $t \in (-\pi, \pi)$.

6. This question is related to a geometric meaning of the Jacobi identity

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Throughout the question, you may find the following identity useful:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

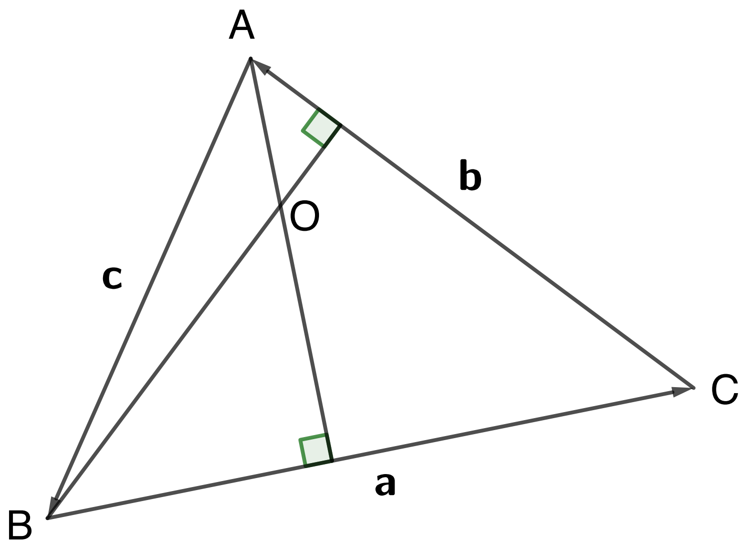

Figure 1: $\triangle ABC$ with two altitudes

As shown in Figure 1, let $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$. Define

$$\mathbf{N_a} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$
$$\mathbf{N_b} = \mathbf{b} \times (\mathbf{c} \times \mathbf{a})$$
$$\mathbf{N_c} = \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$

Let O be the intersection point of the two altitudes passing through A and B respectively.

(a) (*6 points*) Show that $\overrightarrow{OA} \times \mathbf{N_a} + \overrightarrow{OB} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} = \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c}$.
(**Hint:** Express $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of $\overrightarrow{OA}$, $\mathbf{c}$ and $\mathbf{a}$ respectively.)

(b) (*8 points*) Show that $\mathbf{c} \times \mathbf{N_b} + (-\mathbf{b}) \times \mathbf{N_c} = \mathbf{0}$.
(**Hint:** Show that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$ first. Let $\mathbf{u} = \mathbf{c}$, $\mathbf{v} = \mathbf{b}$, $\mathbf{w} = \mathbf{a} \times \mathbf{b}$ and apply Jacobi identity)

(c) (*10 points*) Combining the results from (a) and (b), show that the altitude that passes through C will pass through O.
(**Hint:** Think about the relationship between $\mathbf{N_a}, \mathbf{N_b}, \mathbf{N_c}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$.)

Proof.

(a) Note that

$$\begin{cases} \overrightarrow{OB} = \overrightarrow{OA} + \mathbf{c} \\ \overrightarrow{OC} = \overrightarrow{OA} + \mathbf{c} + \mathbf{a} \end{cases}$$

Then, we have

$$\overrightarrow{OA} \times \mathbf{N_a} + \overrightarrow{OB} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} = \overrightarrow{OA} \times (\mathbf{N_a} + \mathbf{N_b} + \mathbf{N_c}) + \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c}$$

From the Jacobi identity, we have

$$\mathbf{N_a} + \mathbf{N_b} + \mathbf{N_c} = \mathbf{0}$$

and thus

$$\begin{aligned} \overrightarrow{OA} \times \mathbf{N_a} + \overrightarrow{OB} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} &= \overrightarrow{OA} \times \mathbf{0} + \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c} \\ &= \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c} \end{aligned}$$

□

(b) Note that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

It follows that

$$\begin{aligned} \mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{0} \\ \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} &= \mathbf{0} \\ \mathbf{a} \times \mathbf{b} &= -\,\mathbf{a} \times \mathbf{c} \\ &= \mathbf{c} \times \mathbf{a} \end{aligned}$$

Then, the Jacobi identity implies that

$$\begin{aligned} \mathbf{c} \times \mathbf{N_b} + (-\mathbf{b}) \times \mathbf{N_c} &= \mathbf{c} \times \big(\mathbf{b} \times (\mathbf{c} \times \mathbf{a})\big) - \mathbf{b} \times \big(\mathbf{c} \times (\mathbf{a} \times \mathbf{b})\big) \\ &= \mathbf{c} \times \big(\mathbf{b} \times (\mathbf{c} \times \mathbf{a})\big) + \mathbf{b} \times \big((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\big) \\ &= \mathbf{c} \times \big(\mathbf{b} \times (\mathbf{c} \times \mathbf{a})\big) + \mathbf{b} \times \big((\mathbf{c} \times \mathbf{a}) \times \mathbf{c}\big) \\ &= -\,(\mathbf{c} \times \mathbf{a}) \times (\mathbf{c} \times \mathbf{b}) \end{aligned}$$

Furthermore, we also have

$$\begin{aligned} (\mathbf{a} + \mathbf{b} + \mathbf{c}) \times \mathbf{b} &= \mathbf{0} \times \mathbf{b} \\ \mathbf{a} \times \mathbf{b} &= -\,\mathbf{c} \times \mathbf{b} \end{aligned}$$

Together with the previous result, we have

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a} = -\,\mathbf{c} \times \mathbf{b}$$

Thus, it follows that

$$\begin{aligned} \mathbf{c} \times \mathbf{N_b} + (-\mathbf{b}) \times \mathbf{N_c} &= -\,(\mathbf{c} \times \mathbf{a}) \times (\mathbf{c} \times \mathbf{b}) \\ &= (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{0} \end{aligned}$$

□

(c) Combining the results from (a) and (b), we have

$$\begin{aligned}\overrightarrow{OA} \times \mathbf{N_a} + \overrightarrow{OB} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} &= \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c} && \text{(from (a))} \\ &= \mathbf{c} \times \mathbf{N_b} + (-\mathbf{b}) \times \mathbf{N_c} \\ &= \mathbf{0} && \text{(from (b))}\end{aligned}$$

Note that the altitude passes through C must be perpendicular to $\overrightarrow{AB} = \mathbf{c}$ by definition. It remains to show that $\overrightarrow{OC}$ is orthogonal to $\mathbf{c}$, or $\overrightarrow{OC}$ is parallel to $\mathbf{N_c}$ and finish the proof.

- $\overrightarrow{OC}$ is parallel to $\mathbf{N_c}$:

Observe that

$$\langle \mathbf{N_a}, \mathbf{a} \rangle = \langle \mathbf{N_b}, \mathbf{b} \rangle = 0,$$

and from the second useful identity, we have

$$\begin{cases} \mathbf{N_a} &= \langle \mathbf{a}, \mathbf{c} \rangle \, \mathbf{b} - \langle \mathbf{a}, \mathbf{b} \rangle \, \mathbf{c} \\ \mathbf{N_b} &= \langle \mathbf{b}, \mathbf{a} \rangle \, \mathbf{c} - \langle \mathbf{b}, \mathbf{c} \rangle \, \mathbf{a} \end{cases}$$

Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, therefore

$$\mathbf{N_a}, \mathbf{N_b} \in \text{span}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \text{span}(\mathbf{a}, \mathbf{b}) = \text{span}(\mathbf{a}, \mathbf{c}) = \text{span}(\mathbf{b}, \mathbf{c})$$

which means that $\mathbf{N_a}, \mathbf{N_b}$ are coplanar and lying on $\triangle ABC$.

Hence the altitudes passing through A and B are parallel to $\mathbf{N_a}$ and $\mathbf{N_b}$ respectively. Since $\overrightarrow{OA} \perp \mathbf{a}$ and $\overrightarrow{OB} \perp \mathbf{b}$, so $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are parallel to $\mathbf{N_a}$ and $\mathbf{N_b}$ respectively.

Let $\overrightarrow{OA} = p\mathbf{N_a}$ and $\overrightarrow{OB} = q\mathbf{N_b}$, where p, q are constants.

Putting into the (a) and yields

$$\begin{aligned} p\mathbf{N_a} \times \mathbf{N_a} + q\mathbf{N_b} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} &= \mathbf{0} \\ \overrightarrow{OC} \times \mathbf{N_c} &= \mathbf{0} \end{aligned}$$

and which is equivalent to say $\overrightarrow{OC}$ is parallel to $\mathbf{N_c}$.

As $\langle \mathbf{N_c}, \mathbf{c} \rangle = 0$, and $\mathbf{N_c} = -\mathbf{N_a} - \mathbf{N_b}$, hence $\mathbf{N_c}$ is also lying on $\triangle ABC$, and orthogonal to AB. Hence, the altitude passes through C is parallel to $\mathbf{N_c}$ and $\overrightarrow{OC}$. That is, the altitude that passes through C will pass through O. $\square$

Remark. There is an alternative solution for showing $\overrightarrow{OC}$ is orthogonal to $\mathbf{c}$ by Tutor Tommy Lam.

- $\overrightarrow{OC}$ is orthogonal to $\mathbf{c}$:

Combining the results from (a) and (b), we have

$$\begin{aligned}\overrightarrow{OA} \times \mathbf{N_a} + \overrightarrow{OB} \times \mathbf{N_b} + \overrightarrow{OC} \times \mathbf{N_c} &= \mathbf{c} \times \mathbf{N_b} + (\mathbf{c} + \mathbf{a}) \times \mathbf{N_c} && \text{(from (a))} \\ &= \mathbf{c} \times \mathbf{N_b} + (-\mathbf{b}) \times \mathbf{N_c} \\ &= \mathbf{0} && (*)\end{aligned}$$

Denote the normal vector to the plane Π containing $\triangle ABC$ by $\mathbf{N}$.
Then, we have $\mathbf{N}$ is parallel to $\mathbf{b} \times \mathbf{c}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$.
Now, it follows that

$$\left\langle \mathbf{N_a}, \mathbf{a} \right\rangle = \langle \mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \mathbf{a} \rangle = 0$$

$$\left\langle \mathbf{N_a}, \mathbf{N} \right\rangle = \langle \mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \mathbf{N} \rangle = 0 \, .$$

Also, we have $\left\langle \overrightarrow{OA}, \mathbf{a} \right\rangle = 0$ as $\overrightarrow{OA}$ is lying on the altitude passes through A, and $\left\langle \overrightarrow{OA}, \mathbf{N} \right\rangle = 0$ as $\overrightarrow{OA}$ lies on the plane Π .
Therefore, it follows that

$$\overrightarrow{OA} \text{ is parallel to } \mathbf{N_a} \implies \overrightarrow{OA} \times \mathbf{N_a} = \mathbf{0} \, .$$

By the same idea, we also have

$$\overrightarrow{OB} \times \mathbf{N_b} = \mathbf{0} \, .$$

Putting back to (*), we have

$$\mathbf{0} + \mathbf{0} + \overrightarrow{OC} \times \mathbf{N_c} = \mathbf{0} \implies \overrightarrow{OC} \times \mathbf{N_c} = \mathbf{0}$$

Because $\left\langle \mathbf{N_c}, \mathbf{c} \right\rangle = 0$, and $\overrightarrow{OC}$ is parallel to $\mathbf{N_c}$, thus

$$\left\langle \overrightarrow{OC}, \mathbf{c} \right\rangle = 0$$

that is equivalent to say $\overrightarrow{OC}$ is orthogonal to $\mathbf{c}$.
Thus, the altitude that passes through C will pass through O . $\square$

7. (a) Let $\mathbf{v}_n : \mathbb{R}^+ \to \mathbb{R}^2$ be defined by $\mathbf{v}_n(t) = (t^n \cos t, t^n \sin t)$ for any $t \in \mathbb{R}^+$.

i. (*2 points*) Find $\mathbf{v}_n'(t)$.

ii. (*4 points*) Find the arc-length of $\mathbf{v}_n(t)$ from $t = 1$ to $t = 2$ for $n = -1$ and $n = -2$.

(b) Let $\mathbf{u}_{a,b} : \mathbb{R} \to \mathbb{R}^2$ be defined by $\mathbf{u}_{a,b}(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$ for any $t \in \mathbb{R}$.

i. (*2 points*) Find $\mathbf{u}_{a,b}'(t)$.

ii. (*3 points*) Find the arc-length of $\mathbf{u}_{a,b}(t)$ from $t = 0$ to $t = 1$.

Solution.

(a) i. $\mathbf{v}_n'(t) = (-t^n \sin t + nt^{n-1} \cos t, t^n \cos t + nt^{n-1} \sin t)$, for any $t \in \mathbb{R}^+$.

ii. For $n = -1$, we have

$$\mathbf{v}_{-1}'(t) = \left(\frac{-\sin t}{t} - \frac{\cos t}{t^2}, \frac{\cos t}{t} - \frac{\sin t}{t^2} \right)$$

and

$$\begin{aligned}\|\mathbf{v}_{-1}'(t)\| &= \frac{1}{t^2}\sqrt{(t \sin t + \cos t)^2 + (t \cos t - \sin t)^2} \\ &= \frac{1}{t^2}\sqrt{t^2(\sin^2 t + \cos^2 t) + 2t \sin t \cos t - 2t \cos t \sin t + (\cos^2 t + \sin^2 t)} \\ &= \frac{1}{t^2}\sqrt{t^2 + 1}\end{aligned}$$

The arc-length of $\mathbf{v}_{-1}(t)$ from $t = 1$ to $t = 2$ is given by

$$\int_1^2 \|\mathbf{v}_{-1}'(t)\| dt = \int_1^2 \frac{1}{t^2}\sqrt{t^2 + 1}\, dt$$

Putting $t = \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then we have

$$t = 1 \iff \theta = \frac{\pi}{4}$$

and

$$t = 2 \iff \theta = \tan^{-1} 2$$

Hence, the arc-length required is given by

$$\begin{aligned}\int_1^2 \frac{1}{t^2}\sqrt{t^2 + 1}\, dt &= \int_{\frac{\pi}{4}}^{\tan^{-1} 2} \frac{1}{\tan^2 \theta} \sec \theta \cdot \sec^2 \theta d\theta \\ &= \int_{\frac{\pi}{4}}^{\tan^{-1} 2} \sec \theta d\theta + \int_{\frac{\pi}{4}}^{\tan^{-1} 2} \frac{\cos \theta}{\sin^2 \theta} d\theta \\ &= \ln \left| \sec \theta + \tan \theta \right| \Big|_{\frac{\pi}{4}}^{\tan^{-1} 2} - \frac{1}{\sin \theta} \Big|_{\frac{\pi}{4}}^{\tan^{-1} 2} \\ &= \ln(\sqrt{5} + 2) - \ln(\sqrt{2} + 1) + \sqrt{2} - \sqrt{\frac{5}{4}} \\ &= \sqrt{2} - \frac{\sqrt{5}}{2} + \ln(2 + \sqrt{5}) - \ln(1 + \sqrt{2})\end{aligned}$$

Remark. The alternative answer can be given by

$$\sqrt{2} - \frac{\sqrt{5}}{2} + \sinh^{-1} 2 - \sinh^{-1} 1 .$$

For $n = -2$, we have

$$\mathbf{v}'_{-2}(t) = \left(-\frac{\sin t}{t^2} - \frac{2\cos t}{t^3}, \frac{\cos t}{t^2} - \frac{2\sin t}{t^3}\right)$$

and

$$\begin{aligned}\|\mathbf{v}'_{-2}(t)\| &= \frac{1}{t^3}\sqrt{(t\sin t + 2\cos t)^2 + (t\cos t - 2\sin t)^2} \\ &= \frac{1}{t^3}\sqrt{t^2(\sin^2 t + \cos^2 t) + 4t\sin t\cos t - 4t\cos t\sin t + 4(\cos^2 t + \sin^2 t)} \\ &= \frac{1}{t^3}\sqrt{t^2+4}\end{aligned}$$

The arc-length of $\mathbf{v}_{-2}(t)$ from $t = 1$ to $t = 2$ is given by

$$\int_1^2 \|\mathbf{v}'_{-2}(t)\| dt = \int_1^2 \frac{1}{t^3}\sqrt{t^2+4}\, dt$$

Putting $t = 2\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then we have

$$t = 1 \iff \theta = \tan^{-1}\tfrac{1}{2}$$

and

$$t = 2 \iff \theta = \tfrac{\pi}{4}$$

Hence, we have

$$\begin{aligned}\int_1^2 \frac{1}{t^3}\sqrt{t^2+4}\, dt &= \int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \frac{1}{8\tan^3\theta} \cdot 2\sec\theta \cdot 2\sec^2\theta\, d\theta \\ &= \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc^3\theta\, d\theta \\ &= -\frac{1}{2}\csc\theta\cot\theta\Big|_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} - \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc\theta\cot^2\theta\, d\theta \\ &= -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{5}\cdot 2 - \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc^3\theta d\theta + \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc\theta d\theta \\ &= \frac{1}{2}(2\sqrt{5} - \sqrt{2}) - \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc^3\theta d\theta - \frac{1}{2}\ln\left|\csc\theta + \cot\theta\right|\Big|_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \\ &= \frac{1}{2}(2\sqrt{5} - \sqrt{2}) - \frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc^3\theta d\theta - \frac{1}{2}\ln(\sqrt{2}+1) + \frac{1}{2}\ln(\sqrt{5}+2)\end{aligned}$$

This implies that

$$\frac{1}{2}\int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} \csc^3\theta d\theta = \frac{1}{2}\left(\frac{1}{2}(2\sqrt{5} - \sqrt{2}) + \frac{1}{2}\ln(2+\sqrt{5}) - \frac{1}{2}\ln(1+\sqrt{2})\right)$$

Thus, the arc-length required is given by

$$\frac{1}{4}\left(2\sqrt{5} - \sqrt{2} + \ln(2+\sqrt{5}) - \ln(1+\sqrt{2})\right).$$

Remark. If you use hyperbolic substitution, the alternative answer can be given by

$$\frac{1}{4}(2\sqrt{5} - \sqrt{2} + \sinh^{-1}2 - \sinh^{-1}1).$$

(b) i. $\mathbf{u}'_{a,b}(t) = (-ae^{bt}\sin t + abe^{bt}\cos t,\ ae^{bt}\cos t + abe^{bt}\sin t)$, for any $t \in \mathbb{R}$.

ii. From (b)(i), note that

$$\mathbf{u}'_{a,b}(t) = ae^{bt}(-\sin t + b\cos t,\ \cos t + b\sin t)$$

and

$$\begin{aligned}\|\mathbf{u}'_{a,b}(t)\| &= |a|e^{bt}\sqrt{(-\sin t + b\cos t)^2 + (\cos t + b\sin t)^2} \\ &= |a|e^{bt}\sqrt{\sin^2 t + \cos^2 t - b\sin t\cos t + b\cos t\sin t + b^2(\cos^2 t + \sin^2 t)} \\ &= |a|e^{bt}\sqrt{1+b^2}\end{aligned}$$

Thus, the arc-length of $\mathbf{u}_{a,b}(t)$ from $t = 0$ to $t = 1$ is given by

$$\begin{aligned}\int_0^1 \|\mathbf{u}'_{a,b}(t)\| dt &= \int_0^1 |a|e^{bt}\sqrt{1+b^2}\, dt \\ &= |a|\sqrt{1+b^2}\int_0^1 e^{bt} dt \\ &= \begin{cases} |a| & \text{if} \quad b = 0 \\ \frac{|a|}{b}\sqrt{1+b^2}(e^b - 1) & \text{if} \quad b \neq 0 \end{cases}\end{aligned}$$

— *THE END* —